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NOTE ON THE DEFINITIONS OF A VARIABLE.

By DR. G. A. MILLER.

Some mathematical definitions which present no logical difficulties seem at first so unnatural as to cause considerable comment. Among these are the definitions for 'variable number' and for the 'symbol which is a variable'. The natural numbers furnish examples of ideal invariants. Not only are they the same in every country but we think of them as the same throughout the universe. They are certainly included in Sir Oliver Lodge's interesting observation, "Real living arithmetic is the same in any country; and the most important of all is that which must necessarily be the same on any planet."[*]

Even when the concept of numbers has been extended so as to include all real and complex numbers we are in the habit of thinking of them as constants and finite.[†] One of the important uses of numbers is to express a ratio between two quantities of the same kind. One of these quantities may be supposed fixed while the other varies. If we represent this ratio by r it is convenient to speak of r as a variable. No mathematician would hesitate to do this but some would not like to call r a variable number while others do not see any objections to such a usage.

The concept of variable number is analogous to that of variable point. Just as we conceive of some curves as generated by a moving point so we may conceive of the numbers represented by the points of the curve as generated by a number whose elements are the coordinates of the point. In fact, the term point is frequently defined as an aggregate of numbers. The term variable point is so convenient that it is commonly used in all civilized countries,[‡] and some authors who define a variable as a symbol to which we may assign successively different numbers do not hesitate to speak of a variable point.[§]

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[*] *Easy Mathematics, Chiefly Arithmetic;* By Sir Oliver Lodge, 1905, p. 64.

[†] In der Tat haben die Worte unendliche Zahl keinen Sinn, da die Zahl an und feur sich wesentlich endlich ist. Cesaro-Kowalewski, *Elementares Lehrbuch der Algebraischen Analysis und der Infinitesimalrechnung*, 1904, p. 183.

[‡] Cf. G. Peano, *Applicazioni Geometriche de Calcolo Infinitesimale*, 1887, p. 146; Picard et Simart, *Theorie des Fonctions algebriques de deux variables independentes*, 1906, p. 21.

[§] Weber and Wellstein, *Encyklopaedie der Elementar-Mathematik*, Vol. 2, 1905, page 477.

Before the days of Weierstrass writers generally defined the real variable x as a quantity or magnitude which is not constant. All are familiar with such variable magnitudes, for instance, the height of a growing tree. Hence such a definition may at first appear to be a very natural one and it is still employed by a good many writers. From a logical standpoint it seems to present serious difficulties* since the variable x is subjected to the ordinary laws of operation without any inquiry as to the meaning of these operations when applied to the different possible quantities. When the movement towards arithmetizing mathematics began it seemed best to many to replace the term quantity or magnitude by the more definite term number.

While the variable usually stands for any one of a series of numbers yet it is not always desirable to think of it in this way. For instance, in the theory of algebraic numbers the symbols for the variables stand simply for things with which we may operate and have no independent meaning.† In view of this fact many modern writers prefer to define the variable as a symbol to which we may assign various meanings according to the subject under consideration. For instance, in the infinitesimal calculus and function theory the real variable may be defined as a symbol to which we may assign successively different numbers.‡ Whether these numbers constitute all the possible numbers in a given interval or only some of them, such as the integers or multiples of a given number, does not affect the definition.

If it is granted that the real variable may be defined in analytic geometry and calculus as a symbol to which different fixed numbers may be assigned successively, it does not follow directly that this is the most suitable definition for the majority of the students who pursue these courses. The initial concept of symbol carries with it the idea of constancy almost as fully as the initial concept of number does and it is doubtful whether the term variable number offers greater difficulty to the student than the expression the symbol x is a variable. The former is used by such careful writers as Peano, Genocchi-Peano, *Differentialrechnung und Grundzuge der Integralrechnung,* translated into German by Bohlmann and Schepp, 1899, p. 3; and Jordan, *Cours d'Analyse,* Vol. 1, 1893, p. 172.§

It is not to be inferred that those who define the variable as a symbol necessarily imply that the term variable number would be objectionable. In many cases the desire to avoid expressions which may present difficulties *per se* in order to fix the attention upon more important things has doubtless been a more potent motive. The expression the symbol x is a variable is somewhat shorter than the symbol x represents a variable number, and it

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*Cf. Pierpont, *Bulletin of the American Mathematical Society,* Vol. 5, 1899, p. 396.

†In particular cases it is desirable to replace these variables successively by fixed numbers. Cf. Weber, *Lehrbuch der Algebra,* Vol. 2, 1899, p. 568.

‡This definition is adopted by a large number of authors; Cf. Pringsheim, *Encyklopaedie der Mathematischen Wissenschaften,* Vol. 2, p. 8.

§In his *Lecons sur les fonctions de variables reelles,* 1905, Borel frequently speaks of nombre fixe, which implies the existence of others; *e. g.* p. 39.

is more convenient to speak of the variable x than the variable number represented by x. Hence the symbol definition of a variable has a slight advantage with respect to brevity and it makes it easier for those whose desire to avoid the term variable number.

The main object of the preceding remarks is to call attention to the principal elements involved in the different definitions of the term variable, and to give some reasons for the existence of these differences. The teacher is frequently called on to justify his views on points where different usages exist and it is desirable to be able to give references as well as reasons even if the difference appears somewhat trivial. If all numbers were essentially constants I presume most of us would agree with the foot-note quoted from Cesaro to the effect that there is no such thing as an infinite number, for we cannot conceive of any constant number which has such a distinctive character; but it is not so difficult to conceive of a variable which increases without limit and it is convenient to express this concept by saying that the variable is infinite. The ideas of infinite number and variable number are therefore closely related. The existence of the number zero, on the contrary, does not necessarily involve the concept of variability.

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NOTE ON A FORMULA FOR THE SUMMATION OF CERTAIN POWER SERIES.

By HENRY KEMMERLING, Scranton, Pa.

Weierstrass's factor-theorem* is

$$fx = e^{Gx} \prod_{n=1}^{\infty} \left(1 - \frac{x}{a_n}\right) e^{x/a_n + \frac{1}{2}(x/a_n)^2 + [1/(n-1)]\,(x/a_n)^{n-1}},$$

where fx has zeros at the points a_1, a_2, a_3, ...

Taking logarithms and differentiating,

$$\frac{f'x}{fx} = G'x + \sum_{n=1}^{\infty} \left(\frac{1}{x - a_n} + \frac{1}{a_n} + \frac{x}{a_n^2} + \dots \frac{x^{n-2}}{a_n^{n-1}} \right).$$

Then if $\left(\frac{f'x}{fx} - G'x\right)$ is Px, we have

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*Harkness and Morley's *Introduction to Analytic Functions*, p. 199